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Conservation Form of the Equations of Hydrodynamics in Curvilinear Coordinate Systems[1]

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ABSTRACT

In this paper we show how to transform the equations of hydrodynamics so that conservation form is retained in any curvilinear coordinate system.

I. INTRODUCTION

In this paper we shall consider the hydrodynamic equations for perfect fluids in the absence of external and dissipative forces. These equations in integral form may be given as follows:

$$\begin{gathered}\int_{v(t)} \left\{ \frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \mathbf{u}) \right\} dv = 0, \\ \int_{v(t)} \left\{ \frac{\partial (\rho \mathbf{u})}{\partial t} + \operatorname{div}(\rho \mathbf{u}\mathbf{u} + p\mathbf{I}) \right\} dv = 0, \\ \int_{v(t)} \left\{ \frac{\partial E}{\partial t} + \operatorname{div}[\mathbf{u}(E + p)] \right\} dv = 0,\end{gathered} \tag{1}$$

where ρ, $\mathbf{u}$, E, and p are the density, velocity, total energy per unit volume, and pressure, respectively. $\mathbf{I}$ is the identity tensor and $v(t)$ is the time-dependent material volume.

For continuous integrands the integral equations are equivalent to the following differential equations:

$$\begin{gathered}\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \mathbf{u}) = 0, \\ \frac{\partial (\rho \mathbf{u})}{\partial t} + \operatorname{div}(\rho \mathbf{u}\mathbf{u}) + \operatorname{grad} p = 0, \\ \frac{\partial E}{\partial t} + \operatorname{div}[\mathbf{u}(E + p)] = 0.\end{gathered} \tag{2}$$

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In the presence of discontinuities the Rankine–Hugoniot relations which are embedded in the integral formulation need to be appended to the differential equations. For example, in the method of characteristics the shock is considered to be an internal moving boundary. The hydrodynamic equations are integrated up to one side of the discontinuity at which point the Rankine–Hugoniot equations are used to find the conditions on the other side. The characteristic equations may then be used again for the remaining smooth part of the flow. One of the difficulties of this method is that the motion of the discontinuity is not known in advance but is governed by the differential equations and the boundary conditions themselves, i.e., the Rankine–Hugoniot relations.

In a series of papers beginning in 1954 Lax [1] introduced a method for the numerical differencing of Eqs. (3). A consequence of this numerical technique is that it automatically satisfies the Rankine–Hugoniot conditions. In applying the Lax method it is desirable that Eqs. (2) be in what we call here conservation form[4]

$$w_{,t} + f^r{}_{,r} = 0. \tag{3}$$

Equations (2) are automatically in conservation form when Cartesian coordinates are employed. However, in general, they will not be in this form: when other than Cartesian coordinates are employed undifferentiated terms appear and the form (3) is lost. In particular, the conservation of linear momentum contains the divergence of a tensor and it is this term which leads to the appearance of Christoffel symbols of the second kind. Terms of this type, i.e., the centrifugal and coriolis forces arise because of the curvature of the coordinate system. Calculations have been performed by one of us (ELR, Ref. [2]) for the flow around a blunt body using cylindrical and polar coordinates. The equations were written with the Christoffel symbols appearing explicitly.

The purpose of this paper is to transform the equations of hydrodynamics so that the form of Eqs. (3) is retained in any curvilinear coordinate system.

II. Mathematical Preliminaries [3]

Let us consider a field $y_A(x)$ defined over a spatial manifold $\mathcal{M}_3$ and a mapping of $\mathcal{M}_3$ on to itself which maps the point x^r on to the point x'^r where

$$x^r \to x'^r = x'^r(x).$$

[4] In this and following equations latin indices take on the values 1, 2, 3. The Einstein summation convention and the comma notion for differentiation are employed. Thus $f^r{}_{,r} = \partial f^1/\partial x^1 + \partial f^2/\partial x^2 + \partial f^3/\partial x^3$.

The functions $x'^r(x)$ characterize the particular mapping in question. For an infinitesimal mapping,

$$x^r \to x'^r = x^r + \xi^r(x)$$

where the $\xi^r(x)$ are infinitesimal functions of the x^r. Under a mapping the field $y_A(x)$ gets transformed according to

$$y_A(x) \to y'_A(x')$$

where, for the fields we have to consider, $y'_A(x')$ is a linear homogeneous function of the $y_A(x)$. For an infinitesimal mapping,

$$y_A(x) \to y'_A(x') = y_A(x) + \delta y_A(x),$$

where the exact form of $\delta y_A(x)$ depends on the type of field considered. In Table I we have listed the forms of $\delta y_A(x)$ for the types of fields we will encounter in this work.

TABLE I

field	δ (field)
scalar, φ	$\delta\varphi = 0$
contravector, h^r	$\delta h^r = h^s \xi^r{}_{,s}$
covector, k_r	$\delta k_r = -k_s \xi^s{}_{,r}$
cotensor, l_{rs}	$\delta l_{rs} = -l_{rt}\xi^t{}_{,s} - l_{ts}\xi^t{}_{,r}$

In order to characterize the properties of a given tensor field it is necessary to compare its components at a given point of the spatial manifold before and after a mapping. For an infinitesimal mapping this difference, which we designate by $\bar{\delta} y_A(x)$, is given by

$$\begin{aligned} \bar{\delta} y_A(x) &= y'_A(x) - y_A(x) \\ &= \delta y_A(x) - y_{A,r}\xi^r \end{aligned}$$

where we have expanded $y'_A(x)$ about x' and have retained only small quantities of the first order. This latter assumption allows us to drop the prime in the $y'_{A,r}\xi^r$ term.

If $\bar{\delta} y_A(x) = 0$ for a given mapping we say that the mapping is a symmetry of $y_A(x)$. Of particular importance to us here are the symmetries of flat metric g_{rs} of Newtonian space. These symmetries are common to all Newtonian systems and lead to the conservation of linear and angular momentum for these systems.

(Conservation of energy is likewise associated with the time translational symmetry of Newtonian systems.) We will see in the next section that, given the form of the ξ^r that correspond to the symmetries of g_{rs} in an arbitrary coordinate system we can formulate equations of motion in conservation form in that system.

The condition that $\delta g_{rs} = 0$ is

$$\delta g_{rs} = -g_{ru}\xi^u{}_{,s} - g_{us}\xi^u{}_{,r} - g_{rs,u}\xi^u. \tag{4}$$

If we introduce the covector $\xi_r = g_{rs}\xi_s$ we can rewrite Eq. (4) in the form

$$\xi_{r;s} + \xi_{s;r} = 0 \tag{5}$$

where the semicolon denotes covariant differentiation, i.e.,

$$\xi_{r;s} = \xi_{r,s} - \left\{ {u \atop r \quad s} \right\} \xi_u$$

with the Christoffel symbol $\{_r{}^u{}_s\}$ given by

$$\left\{ {u \atop r \quad s} \right\} = \frac{1}{2} g^{uv}(g_{rv,s} + g_{vs,r} - g_{rs,v}).$$

Equation (5) is known as Killing's equation and any vector ξ_r that satisfies it as a Killing vector [4].

III. Transformation of Equations[5]

The existence of equations that govern the behavior of a continuum system in conservation form is a consequence of the underlying symmetries of the system. It is well-known that, corresponding to each such symmetry, there exists a constant of the motion which for a continuum system will be of the form $C = \int w dV$ where w is a scalar density. Since $dC/dt = 0$, it follows that $\partial w/\partial t$ must differ from zero by at most a divergence:

$$\partial w/\partial t + f^r{}_{,r} = 0.$$

For each symmetry we can therefore expect a conservation law of the form (3). A rigorous proof of these assertions is to be found in Noether's theorem. The essential point for us is that the symmetries of a physical system are intrinsic to it and do not

[5] The main result of this section has been proved in another way by A. Lapidus in his doctoral dissertation [5].

depend on the existence of any particular choice of coordinate system. It follows therefore that, since the equations of hydrodynamics are in the form (3) in a Cartesian coordinate system, it must be possible to express them in this form in an arbitrary coordinate system. In this section we shall express Eqs. (2) in conservation form for arbitrary curvilinear coordinate systems

The conservation of mass and energy in Cartesian coordinates may be expressed in the following form:

$$w_{,t} + f^r{}_{,r} = 0.$$

In curvilinear cordinates, partial differentiation is replaced by covariant differentiation.

$$w_t + f^r{}_{;r} = 0$$

Equation (6) is no longer of the form (3). However, if we multiply Eq. (6) by $g^{1/2}$, where g is the determinant of g_{rs} and take account of the fact that $g_{;\mu} = 0$, we arrive at the desired form:

$$(g^{1/2}w)_{,t} + (g^{1/2}f^r)_{,r} = 0. \tag{7}$$

It is important to recognize that the f^r are the tensor components and not the physical components. A clear discussion of this distinction is given in Truesdell's article [6].

Let us now consider the equations expressing the conservation of linear momentum. In Cartesian coordinates these equations have the form

$$w^r{}_{,t} + f^{rs}{}_{,s} = 0 \tag{8}$$

where $f^{rs} = f^{sr}$. Again, in curvilinear coordinates, ordinary differentiation gets replaced by covariant differentiation and we have

$$w^r{}_{,t} + f^{rs}{}_{;s} = 0. \tag{9}$$

If we multiply this equation by $g^{1/2}$ times a vector ξ_r and sum over r, the linear momentum equation has the form (3) if and only if

$$\xi_{r;s} + \xi_{s;r} = 0. \tag{10}$$

For vectors that satisfy (10) we have

$$(g^{1/2}\xi_r w^r)_{,r} + (g^{1/2}\xi_r f^{rs})_{,s} = 0. \tag{11}$$

In the case of a flat three-dimensional metric, Killing's equation admits six independent solutions corresponding to three translations and three rotations. For perfect fluids it can be shown that the conservation of angular momentum is a consequence of Eqs. (1) and the three rotational Killing vectors may be ignored [7]. In Appendix A we give the form of the translational Killing vectors and the hydrodynamic equations in cylindrical and spherical coordinates.

It is possible to rewrite Eqs. (11) in a form that agrees with those of Lapidus. Let $\xi_r^{(i)}$ be the three orthogonal Killing vectors corresponding to translations. If we define

$$\bar{w}^i = g^{1/2}\xi_r^{(i)}w^r$$

and

$$f^{ij} = g^{1/2}\xi_r^{(i)}\xi_s^{(j)}f^{rs},$$

then Eq. (11) can be put into the form

$$\frac{\partial}{\partial t}\bar{w}^i + (\xi_{(j)}^s f^{ij})_{,s} = 0, \tag{12}$$

where $\xi_{(j)}^s$ is the reciprocal matrix to $\xi_s^{(j)}$. Equations (12) are the Lapidus equations. It follows from the form of the Killing vectors in Cartesian coordinates that $\bar{w}^i$ and f^{ij} are the Cartesian components of these quantities as functions of the curvilinear coordinates.

APPENDIX A1. Cylindrical Coordinates (r, θ, z)

The three translational Killing vectors in Cartesian coordinates are

$$\begin{aligned}\xi_1 &= (1, 0, 0),\\ \xi_2 &= (0, 1, 0),\\ \xi_3 &= (0, 0, 1).\end{aligned}$$

The form of these vectors in curvilinear coordinates may be found by transforming these Cartesian vectors. Listed below are the appropriate Killing vectors for cylindrical coordinates.

$$\begin{aligned}\xi_1 &= (\cos\theta, -r\sin\theta, 0)\\ \xi_2 &= (\sin\theta, r\cos\theta, 0)\\ \xi_3 &= (0, 0, 1).\end{aligned}$$

The f^r and f^{rs} are given as

$$f^1 = \frac{u^r}{(g_{11})^{1/2}}, \qquad f^2 = \frac{u^\theta}{(g_{22})^{1/2}}, \qquad f^3 = \frac{u^z}{(g_{33})^{1/2}},$$

where u^r, u^θ, and u^z are the physical components of velocity.

$$f^{rs} = \frac{\rho u^r u^s}{(g_{rr})^{1/2}(g_{ss})^{1/2}} + \frac{\rho \delta^{rs}}{(g_{rr})^{1/2}(g_{ss})^{1/2}},$$

where

$$g_{rs} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The equation of continuity is

$$\{r\rho\}_t + \{r\rho u^r\}_r + \{\rho u^\theta\}_\theta + \{r\rho u^z\}_z = 0.$$

The equations of motion are:

r component:

$$\begin{aligned} &\{r\rho(u^r \cos\theta - u^\theta \sin\theta)\}_t + \{r\cos\theta(\rho u^r u^r + p) - r\rho u^r u^\theta \sin\theta\}_r \\ &\quad + \{\rho u^r u^\theta \cos\theta - \sin\theta(\rho u^\theta u^\theta + p)\}_\theta + \{r\rho u^r u^z \cos\theta - r\rho u^\theta u^z \sin\theta)\}_z = 0; \end{aligned}$$

θ component:

$$\begin{aligned} &\{r\rho(u^r \sin\theta + u^\theta \cos\theta)\}_t + \{r\sin\theta(\rho u^r u^r + p) + r\rho u^r u^\theta \cos\theta\}_r \\ &\quad + \{\rho u^r u^\theta \sin\theta + \cos\theta(\rho u^\theta u^\theta + p)\}_\theta + \{r\rho u^r u^z \sin\theta + r\rho u^\theta u^z \cos\theta\}_z = 0; \end{aligned}$$

z component:

$$\{r\rho u^z\}_t + \{r\rho u^r u^z\}_r + \{\rho u^\theta u^z\}_\theta + \{r(\rho u^z u^z + p)\}_z = 0.$$

The energy equation is

$$\begin{aligned} &\{rE\}_t + \left\{r\left[\gamma u^r E - (\gamma - 1)\frac{\rho}{2}(u^r u^r u^r + u^r u^\theta u^\theta + u^r u^z u^z)\right]\right\}_r \\ &\quad + \left\{\gamma u^\theta E - (\gamma - 1)\frac{\rho}{2}(u^r u^r u^\theta + u^\theta u^\theta u^\theta + u^\theta u^z u^z)\right\}_\theta \\ &\quad + \left\{r\left[\gamma u^z E - (\gamma - 1)\frac{\rho}{2}(u^r u^r u^z + u^\theta u^\theta u^z + u^z u^z u^z)\right]\right\}_z = 0, \end{aligned}$$

γ is the ratio of specific heats.

APPENDIX A2. Spherical Coordinates (r, θ, φ)

The Killing vectors in spherical coordinates are

$$\begin{aligned}
\xi_1 &= (\sin\theta\cos\varphi,\ r\cos\theta\cos\varphi,\ -r\sin\theta\sin\varphi),\\
\xi_2 &= (\sin\theta\sin\varphi,\ r\cos\theta\sin\varphi,\ r\sin\theta\cos\varphi),\\
\xi_3 &= (\cos\theta,\ -r\sin\theta,\ 0).
\end{aligned}$$

The f^r and f^{rs} are given by

$$f^1 = \frac{u^r}{(g_{11})^{1/2}}, \qquad f^2 = \frac{u^\theta}{(g_{22})^{1/2}}, \qquad f^3 = \frac{u^\varphi}{(g_{33})^{1/2}},$$

$$f^{rs} = \frac{\rho u^r u^s}{(g_{rr})^{1/2}(g_{ss})^{1/2}} + \frac{\rho\delta^{rs}}{(g_{rr})^{1/2}(g_{ss})^{1/2}},$$

where

$$g^{rs} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2\theta \end{pmatrix}.$$

The equation of continuity is

$$\{r^2\rho\sin\theta\}_t + \{r^2\rho u^r\sin\theta\}_r + \{r\rho u^\theta\sin\theta\}_\theta + \{r\rho u^\varphi\}_\varphi = 0.$$

The equations of motion are:
r component:

$$\begin{aligned}
&\{r^2\sin\theta[\rho u^r\sin\theta\cos\varphi + \rho u^\theta\cos\theta\cos\varphi - \rho u^\varphi\sin\varphi]\}_t\\
&\quad + \{r^2\sin\theta[\sin\theta\cos\varphi(\rho u^r u^r + p) + \rho u^r u^\theta\cos\theta\cos\varphi - \rho u^r u^\varphi\sin\varphi]\}_r\\
&\quad + \{r\sin\theta[\rho u^r u^\theta\sin\theta\cos\varphi + \cos\theta\cos\varphi(\rho u^\theta u^\theta + p) - \rho u^\theta u^\varphi\sin\theta\sin\varphi]\}_\theta\\
&\quad + \{r\sin\theta[\rho u^r u^\varphi\cos\varphi + \rho u^\theta u^\varphi\cot\theta\cos\varphi - \operatorname{cosec}\theta\sin\varphi(\rho u^\varphi u^\varphi + p)\}_\varphi = 0;
\end{aligned}$$

θ component:

$$\begin{aligned}
&\{r^2\sin\theta[\rho u^r\sin\theta\sin\varphi + \rho u^\theta\cos\theta\sin\varphi + \rho u^\varphi\cos\varphi]\}_t\\
&\quad + \{r^2\sin\theta[\sin\theta\sin\varphi(\rho u^r u^r + p) + \rho u^r u^\theta\cos\theta\sin\varphi + \rho u^r u^\varphi\cos\varphi]\}_r\\
&\quad + \{r\sin\theta[\rho u^r u^\theta\sin\theta\sin\varphi + \cos\theta\sin\varphi(\rho u^\theta u^\theta + p) + \rho u^\theta u^\varphi\cos\varphi]\}_\theta\\
&\quad + \{r\sin\theta[\rho u^r u^\varphi\sin\varphi + \rho u^\theta u^\varphi\cot\theta\sin\varphi + \operatorname{cosec}\theta\cos\varphi(\rho u^\varphi u^\varphi + p)]\}_\varphi = 0;
\end{aligned}$$

φ component:

$$\{r^2 \sin\theta[\rho u^r \cos\theta - \rho u^\theta \sin\theta]\}_t + \{r^2 \sin\theta[\cos\theta(\rho u^r u^r + p) - \rho u^r u^\theta \sin\theta]\}_r$$
$$+ \{r \sin\theta[\rho u^r u^\theta \cos\theta - \sin\theta(\rho u^\theta u^\theta + p)]\}_\theta$$
$$+ \{r \sin\theta[\rho u^r u^\varphi \cot\theta - \rho u^\theta u^\varphi]\}_\varphi = 0.$$

The energy equation is

$$\{r^2 E \sin\theta\}_t + \left\{r^2 \sin\theta \left[\gamma u^r E - (\gamma - 1)\frac{\rho}{2}(u^r u^r u^r + u^r u^\theta u^\theta + u^r u^\varphi u^\varphi)\right]\right\}_r$$
$$+ \left\{r \sin\theta \left[\gamma u^\theta E - (\gamma - 1)\frac{\rho}{2}(u^r u^r u^\theta + u^\theta u^\theta u^\theta + u^\theta u^\varphi u^\varphi)\right]\right\}_\theta$$
$$+ \left\{r \left[\gamma u^\varphi E - (\gamma - 1)\frac{\rho}{2}(u^r u^r u^\varphi + u^\theta u^\theta u^\varphi + u^\varphi u^\varphi u^\varphi)\right]\right\}_\varphi = 0.$$

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